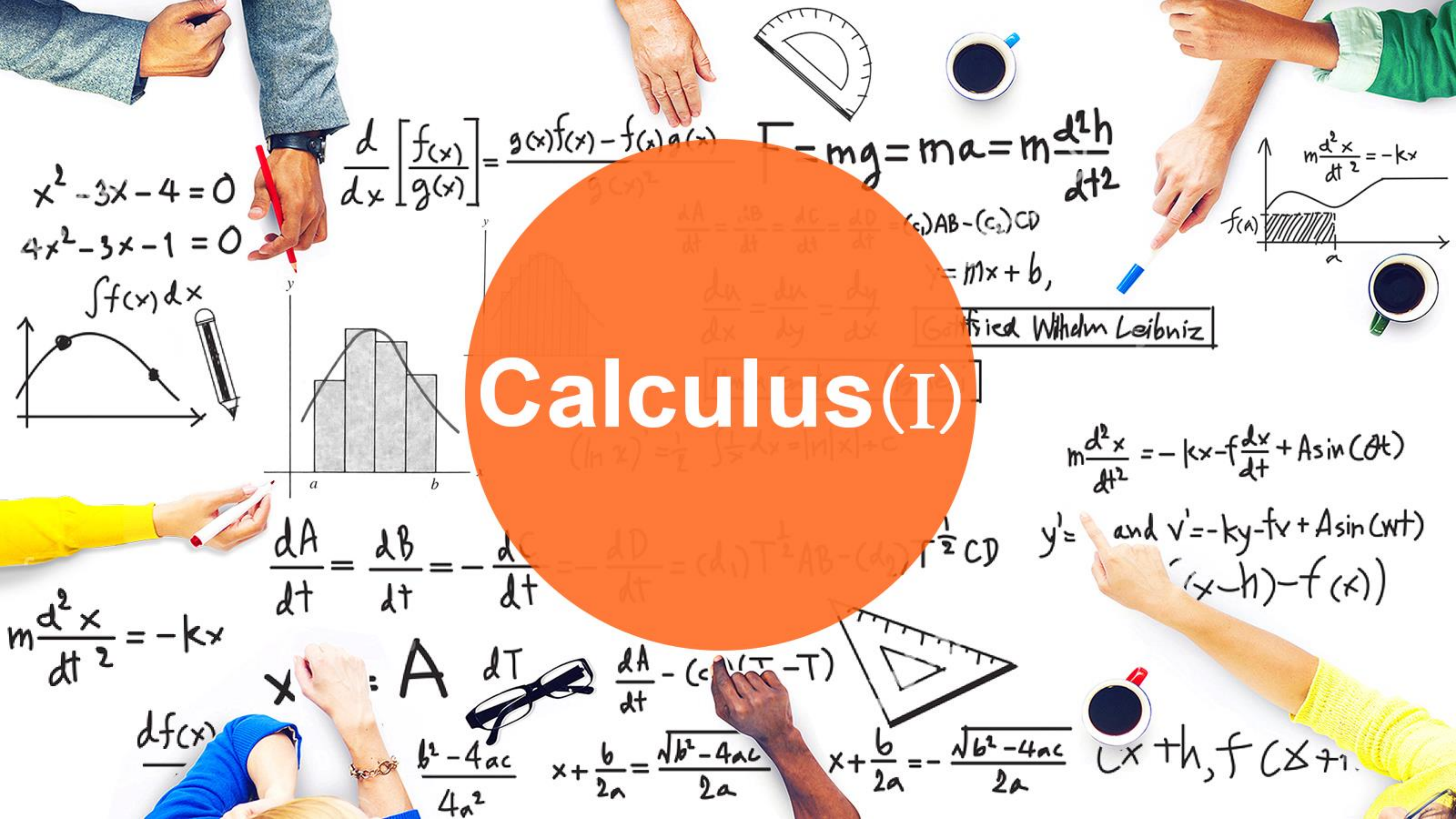
Calculus(I)



Improper Integrals (I):

Infinite Limits of Integration

Lecturer: Xue Deng

Problem Introduction

Proper Integral: Integration interval $[a, b]$ is finite.

extend

How to do? By the limit of proper integral!

Improper Integral: Integration interval $[a, +\infty]$ or $(-\infty, b]$ or $(-\infty, +\infty)$ is infinite.

Problem Introduction

In the definition of $\int_a^b f(x)dx$ it was assumed that the interval $[a,b]$ was finite

However, in many applications in physics, economics, and probability:

We wish to allow a or b (or both) to be $+\infty$ or $-\infty$

So, we must find a way to give meaning to symbols like

$$\int_0^{+\infty} \frac{1}{1+x^2}dx, \int_{-\infty}^{-1} xe^{-x^2}dx, \int_{-\infty}^{+\infty} x^2e^{-x^2}dx$$

These integrals are called **improper integrals** with infinite limits.

Definition 1

One Infinite Limit Consider the function $f(x) = xe^{-x}$

It makes perfectly good sense to ask for $\int_0^1 xe^{-x}dx$ or $\int_0^2 xe^{-x}dx$,

or indeed for $\int_0^b xe^{-x}dx$ where b is any positive number.

To give meaning to $\int_0^{+\infty} xe^{-x}dx$ where b is positive infinity.

we begin by integrating from 0 to an arbitrary upper limit, say b

Definition 1

By parts, we have

$$\int_0^b xe^{-x}dx = \left[-xe^{-x}\right]_0^b - \int_0^b (-e^{-x})dx = 1 - e^{-b} - be^{-b}$$

If we let $b \to +\infty$

$$\int_0^{+\infty} xe^{-x}dx = \lim_{b\to+\infty}\int_0^b xe^{-x}dx = \lim_{b\to+\infty}(1 - e^{-b} - be^{-b}) = 1$$

Definition 2

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx$$

$$\int_{a}^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_{a}^{b} f(x)dx$$

(I) If the limits on the right exist and have finite values:

the corresponding improper integrals **converge** and have those values.

(II) If the limits on the right don't exist:

the integrals are said to **diverge**.

Definition

Rules: By N—L formula If $F'(x) = f(x)$, we have

$$\int_a^{+\infty} f(x)\mathrm{d}x = F(x)\Big|_a^{+\infty} = F(+\infty) - F(a),$$

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = F(x)\Big|_{-\infty}^{b} = F(b) - F(-\infty),$$

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = F(x)\Big|_{-\infty}^{+\infty} = F(+\infty) - F(-\infty).$$

$$F(+\infty) = \lim_{x\to+\infty} F(x).$$

$$F(-\infty) = \lim_{x\to-\infty} F(x).$$

Example 1

$$\int_a^{+\infty} f(x)\mathrm{d}x = F(x)\Big|_a^{+\infty} = F(+\infty) - F(a),$$

Find the value of $\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2} \sin\frac{1}{x}\,\mathrm{d}x$

$$= -\int_{\frac{2}{\pi}}^{+\infty} \sin\frac{1}{x}\,\mathrm{d}\left(\frac{1}{x}\right)$$

$$= \left[\cos\frac{1}{x}\right]_{\frac{2}{\pi}}^{+\infty}$$

$$= \lim_{x\to+\infty} \cos\frac{1}{x} - \cos\frac{\pi}{2}$$

$$= 1.$$

Example 2

$$\int_{-\infty}^{+\infty} f(x)\,dx = F(x)\Big|_{-\infty}^{+\infty} = F(+\infty) - F(-\infty).$$

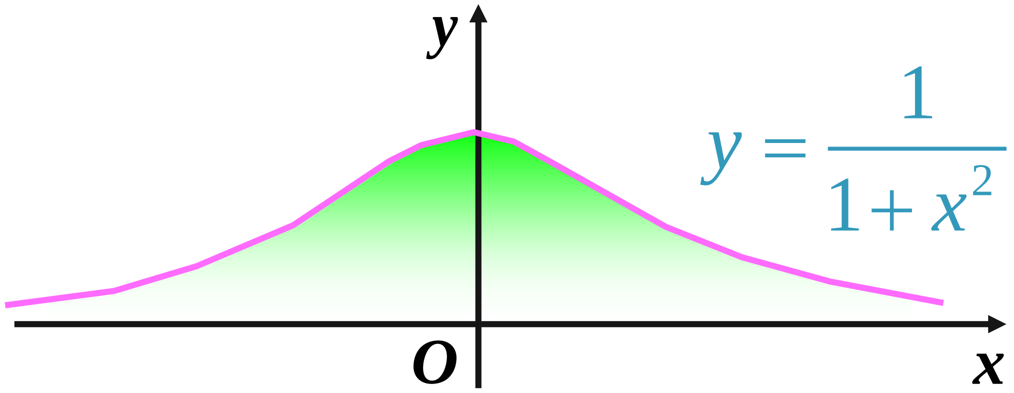
Find the value of $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

$$= \left[\arctan x\right]_{-\infty}^{+\infty}$$

$$= \lim_{x\to+\infty} \arctan x - \lim_{x\to-\infty} \arctan x$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Geometrical meaning of improper integral value



Example 3

Prove the improper integral $\int_a^{+\infty} e^{-px}\mathrm{d}x$,

when $p > 0$, converges; when $p < 0$, diverges.

$$\int_a^{+\infty} e^{-px}\mathrm{d}x = -\frac{1}{p}\int_a^{+\infty} e^{-px} d(-px) = \left[-\frac{e^{-px}}{p}\right]\Bigg|_a^{+\infty}$$

$$= \begin{cases} \dfrac{e^{-ap}}{p}, & \text{when } p > 0, \\ \infty, & \text{when } p < 0. \end{cases}$$

when $p > 0$ converges;

when $p < 0$ diverges.

Summary (Remember & Understand)

If $F(x)$ is the original function of the continuous function $f(x)$.

$$\int_a^{+\infty} f(x)\mathrm{d}x = F(x)\Big|_a^{+\infty} = F(+\infty) - F(a),$$

$$F(+\infty) = \lim_{x\to+\infty} F(x).$$

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = F(x)\Big|_{-\infty}^{b} = F(b) - F(-\infty),$$

$$F(-\infty) = \lim_{x\to-\infty} F(x).$$

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = F(x)\Big|_{-\infty}^{+\infty} = F(+\infty) - F(-\infty).$$

Questions and Answers

Find the value of $\int_{e}^{+\infty} \frac{1}{x\ln^{2} x} dx.$

$$= \int_{e}^{+\infty} \frac{1}{\ln^{2} x} d(\ln x)$$

$$= -\frac{1}{\ln x}\Big|_{e}^{+\infty}$$

$$= \lim_{x\to+\infty} (-\frac{1}{\ln x}) - (-\frac{1}{\ln e})$$

$$= 1.$$

Questions and Answers

The area of locating below the curve $y = xe^{-x}\ (0 \le x < +\infty)$, and above the x axis.

$$A = \int_0^{+\infty} xe^{-x}\,\mathrm{d}x$$

$$= -\int_0^{+\infty} x\,\mathrm{d}e^{-x}$$

$$= -\left[xe^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-x}\,\mathrm{d}x\right]$$

$$= 1.$$

Questions and Answers

Prove the improper integral $\int_1^{+\infty} \frac{1}{x^p} dx$,

when $p > 1$, converges, when $p \le 1$, diverges.

(1) $p = 1$, $\int_1^{+\infty} \frac{1}{x^p} dx = \int_1^{+\infty} \frac{1}{x} dx = [\ln x]_1^{+\infty} = +\infty$

(2) $p \ne 1$, $\int_1^{+\infty} \frac{1}{x^p} dx = \left[\frac{x^{1-p}}{1-p}\right]\Bigg|_1^{+\infty} = \begin{cases} +\infty, & p < 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$

when $p > 1$ converges, and $\int_1^{+\infty} \frac{1}{x^p} dx = \frac{1}{p-1}$;

when $p \le 1$ diverges.

Infinite Limits of Integration